Milan j. math. 76 (2008), 93–123

1424-9286/010093-31, *published online* 14.3.2008
DOI 10.1007/s00032-008-0086-z

Milan Journal of Mathematics

Néron Models and Formal Groups

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Abstract. We show that formal groups can be used to simplify the construction of Néron models. Also we give a new proof of the stable reduction theorem for abelian varieties.

Mathematics Subject Classification (2000). 14F30.

Keywords. p-adic period domains, p-divisible groups.

1. Introduction

Néron models ([6]) are an important tool in arithmetic geometry. Their construction has been outlined in modern terminology by M.Artin ([1]) and by Bosch/Lütkebohmert/Raynaud ([2]). The main purpose of this note is to demonstrate that the use of invariant differential operators and formal groups permits some simplifications. Namely we use the local theory to construct the formal group associated to the Néron-model, which then allows to simplify the global steps. More precisely we follow [2] Ch.3 in the construction of weak Néron-models, then construct directly the formal group associated to the Néron model, and finally use it to to bypass many of the difficulties in [2] Ch. 5. In addition we use the occasion to present some slight generalisation of the construction to non proper and non commutative groupschemes. Here the main idea is to consider the Néron extension property not for all Néron-points but for a bounded subgroup of them, thus avoiding the complications in [2], Ch.10. Finally we give a new proof of the stable reduction theorem which uses neither Jacobians nor the Weil-conjectures.

These notes are an extended version of material first presented at my graduate course at Bonn University. I thank the referee for his very attentive reading.

Suppose V is a discrete valuation-ring, $\pi \in V$ a generator of its maximal ideal, K its field of fractions and

$$k = V/\pi V$$

the residue field. We consider valuation-rings $V' \supset V$ such that π is still a uniformiser in V', and such that the residue field

$$k' = V'/(\pi)$$

is separable over k. Equivalently V' is formally smooth over V. For a scheme X_K over K define a Néron-point of X as a point in $X(K')$, for some $V' \subset K'$ as above. We need the notion of base-change for Néron-points:

If

$$V_1 \supset V$$

denotes an extension of discrete valuation-rings, with $\pi \in (\pi_1)$, and if V' is formally smooth over V, consider the ring

$$R = V' \otimes_V V_1.$$

Its quotient modulo (π_1) is isomorphic to $k' \otimes_k k_1$ and is reduced. Each of its minimal primes defines a prime

$$\mathfrak{p} \subset R$$

such that in the localisation $R_\mathfrak{p}$ the maximal ideal is generated by π_1 which is a non zerodivisor. The π_1-adic completion of $R_\mathfrak{p}$ is a discrete valuation-ring V_1' which is formally smooth over V_1. It would in effect suffice to divide $R_\mathfrak{p}$ by the intersection of all ideals (π_1^n). Furthermore as V' and V_1 inject into V_1' a K'-point of X_K induces a K_1'-point of the base-extension X_{K_1}. We define the base-change of a family of Néron-points of X_K as the family of Néron-points of X_{K_1} obtained this way, for all possible choices of $\mathfrak{p}$. Thus the base-change of one Néron-point is in general not such a point but a family of them. The reason for this choice is the following:

Suppose X_K extends to a separated V-scheme X and we have a family of Néron-points of X_K which extend to integral points

$$x \in X(V').$$

If there images in the special fibre X_k are scheme-theoretically dense, then this remains so after base-change.

Now assume G_K is a smooth separated groupscheme over K. For two Néron-points x_1, x_2 define a product by applying base-change (to V_1' or V_2') and forming the usual product. Note that this gives us a family of points, not just one unique product. Now assume given a set $\mathcal{E}$ of Néron-points which is "open" and "bounded" (definitions follow below) and closed under inverses and products. The latter means that for any two points in $\mathcal{E}$ any of their products also lies in $\mathcal{E}$. Then we have

Theorem 1. *There exists a unique smooth separated group-scheme G over V extending G_K such that*

i) *Any $x' \in G_K(K')$ in $\mathcal{E}$ extends to an element in $G(V')$.*
ii) *The reductions of these points (i.e. the values on k') are Zariski-dense in the special fibre G_k.*

Furthermore if S is a smooth V-scheme together with a K-map $S_K \to G_K$ such that the elements of $\mathcal{E}$ which lift to V'-points of S have Zariski-dense reductions in the special fibre S_k, then the generic map $S_K \to G_K$ extends to $S \to G$.

With this terminology we can also define "open": It means that there exists an S as above, with non empty S_k, together with a smooth dominant map

$$S_K \to G_K,$$

such that the reductions of elements of $\mathcal{E}$ which lift to V'-points of S are Zariski-dense in S_k. Also "bounded" means that there exists a V-scheme X of finite type, together with a map

$$X_K \to G_K,$$

such that all elements of $\mathcal{E}$ extends to V'-points of X. We remark that we may assume that X is quasiprojective and

$$X_K = G_K :$$

Embed (open) G_K into a Y which is projective over V (use [7] for quasiprojectivity of G_K). We may replace X by the closure of X_K in $X \times Y$ and assume that X maps to Y. If

$$Z \subset Y$$

denotes the closure of $Y_K - G_K$ the fibered product $X \times_Y Z$ is supported in the special fibre of X, thus its structuresheaf is annihilated by some power π^r. Blowing up in Y the sum of (π^{r+1}) and the ideal of Z defines a new

$\tilde{Y}$ and a new $\tilde{Z}$ (which is in fact isomorphic to Z). Then all elements of $\mathcal{E}$ extend to V'-points of $\tilde{Y} - \tilde{Z}$.

We finish by quoting some results from [7]. If G is a smooth separated algebraic groupscheme over a field K and $\mathcal{L}$ a line-bundle on G then some positive power $\mathcal{L}^{\otimes n}$ satisfies the theorem of the cube in the following sense:

Lemma 2. *Denote by G° the connected component of the identity. Then on $G^\circ \times G^\circ \times G$ the alternating product of the pullbacks $m_I^*(\mathcal{L}^{\otimes n})$ is canonically trivial.*

Here m_I denotes the partial multiplications (in a fixed order) onto G, and "canonical" refers to the fact that this alternating product is already trivial on the subschemes where one of the factors is the identity element. For the proof we may extend K, assume that

$$G = G^\circ,$$

and quote [7], Th. IV, 2.6.

The lemma implies as usual that G is quasiprojective: Chose an open dense affine $U \subset G$. Then

$$D = G - U$$

is a divisor and

$$\mathcal{L} = \mathcal{O}(D)$$

is ample.

2. Review of Néron desingularisation

Here we have nothing new to say so we just recall what is known ([2], Ch. 3): Suppose X is a quasiprojective V-scheme with X_K smooth of relative dimension d, and $\mathcal{E}$ a set of Néron-points of X_K which extend to V'-points of X. Then we can find a sequence of blow-ups

$$\tilde{X} \to X$$

of subschemes of the special fibre such that all elements of $\mathcal{E}$ define V'-points of the smooth locus of $\tilde{X}$:

The d-th Fitting-ideal

$$\mathcal{I}_d \subset \mathcal{O}_X$$

of the coherent sheaf $\Omega_{X/V}$ is locally generated by the minors of a suitable size in a presentation of $\Omega_{X/V}$. It defines the closed subscheme of X where $\Omega_{X/V}$ cannot be locally generated by d elements, and this subscheme is supported in the special fibre. Thus $\mathcal{I}_d$ contains a power of π. For a V'-point x of X the pullback of $\mathcal{I}_d$ to V' is generated by $\pi'^{,l}$ with l the length of the torsion in $x^*(\Omega_{X/V})$. l vanishes iff $\Omega_{X/V}$ is locally free of rank d near x, which means that x lies in the smooth locus. Furthermore l is bounded above by a constant independant of x.

Now one uses induction first on the maximal value of l and then on the dimension of the scheme-theoretic closure in X_k of the reductions of the elements $\mathcal{E}$. Namely if Y is this scheme-theoretic closure Y is geometrically reduced. Remove a Y' of smaller dimension such that $Y - Y'$ is smooth over k, and such that $\Omega_{X/V}$ is locally free over $Y - Y'$. If

$$\mathcal{E}' \subset \mathcal{E}$$

denotes the subset of Néron-points specialising in Y' we may by induction modify X by blow-ups centered over Y' and assume that the elements of $\mathcal{E}'$ specialise in the smooth locus of X. In this new model denote by Y" the closure of the union of those irreducible components of $Y - Y'$ where $\Omega_{X/V}$ has rank $> d$. Then Y" lies in the locus defined by the d-th Fitting-ideal and this is disjoint from the smooth locus. Thus blowing-up Y" does not affect the points in $\mathcal{E}'$. On the other hand the points in $\mathcal{E} - \mathcal{E}'$ with positive l-invariant specialise in

$$Y\text{"} \cap (Y - Y'),$$

and by a local calculation blowing-up Y" reduces their l-invariant.

Now we want to improve this result: We assume that X_K is geometrically irreducible, that $\Omega^d_{X_K/K}$ is trivial generated by an everywhere nonzero d-form α, and that there exists a smooth V-scheme S, S_k non-empty, together with a smooth map

$$S_K \to X_K$$

such that there exist a family $\mathcal{F}$ of V'-points of S whose reductions are dense in S_k and which induce on X_K Néron points in $\mathcal{E}$ (our openness condition).

Choose a separated V-scheme X with general fibre X_K such the elements of $\mathcal{E}$ extend to V'-points of X. Then the generic map extends to

$$S \to X,$$

after removing from S a closed nowhere dense subset of S_k (or of codimension ≥ 1 in S_k, or of codimension ≥ 2 in S). Namely the closure of the graph of the generic map is a closed subscheme of $S \times X$ which is still flat over V, of the same relative dimension as S. Its projection to S is then flat and quasi-finite after removing a thin subset of S_k as above. By Zariski's main theorem it then is an open immersion into a finite S-scheme. From the general fibre we see that this finite S-scheme is S, so the projection is an open immersion. The V'-points in $\mathcal{F}$ contained in our new (slightly smaller) S still satisfy the density condition and give points in the graph-closure. So we may replace S by this closure.

Secondly we may multiply α by a suitable power of π and assume that it extends to a global section of $\Omega^d_{X/V}$. Then on the smooth locus X^{sm} α becomes a section of the line-bundle $\Omega^d_{X/V}$ whose divisor is a linear combination of irreducible components

$$Y \subset X_k^{sm}.$$

If an element of $\mathcal{F}$ induces a V'-point of X^{sm} specialising in Y the pullback of α to S is a regular section of the vectorbundle $\Omega^d_{S/V}$ which is everywhere nonzero on S_K. Thus its coefficients in a local basis generate a coherent ideal in $\mathcal{O}_S$ which contains a power π^r. Thus the pullback of α is not divisible by π^{r+1}, and the coefficient of Y in the divisor of α is $\leq r$. This bound r is independant of the choice of X as long as α extends to a regular section of $\Omega^d_{X/V}$. Especially we may replace X by blow-ups.

Now we claim that we can find a model X such that all elements of $\mathcal{E}$ extend to integral points of X^{sm}, that for some integer m all irreducible components

$$Y \subset X_k^{sm}$$

have coefficient $\geq m$ in the divisor of α, and that the coefficient is m for some component Y such that their specialisations are Zariski-dense in Y:

First choose some quasiprojective model X such that all elements of $\mathcal{E}$ extend to integral points. Multiply α by a power of π to make it regular on X. In the following new models will be obtained by blow-ups, so α will remain regular on them. Then apply Néron desingularisation to make all points specialise in X^{sm}. Remove the closure of all irreducible components

$$Y \subset X_k^{sm}$$

which do not contain specialisations of elements of $\mathcal{E}$. Finally apply decreasing induction over the minimal coefficient in $div(\alpha)$ of irreducible components

$$Y \subset X_k^{sm}.$$

We know that this number is bounded by r.

Suppose that for a component

$$Y \subset X_k^{sm}$$

the specialisations of elements of $\mathcal{E}$ in it lie in a proper subscheme Y'. We may assume that these specialisations are scheme-theoretic dense in Y', so Y' is geometrically reduced and generically smooth over k. Remove a closed subscheme

$$Y" \subset Y'$$

such that the complement is smooth. Then consider the elements of $\mathcal{E}$ specialising in $Y"$. Their scheme-theoretic closure is geometrically reduced and we may continue the procedure. This way we obtain a finite list of proper subschemes

$$Y'_\mu \subset Y$$

such that any element of $\mathcal{E}$ specialising in Y specialises in the smooth locus of some Y'_μ.

If for some component Y with minimal coefficient m in $div(\alpha)$ the specialisations of points in $\mathcal{E}$ are dense we are done. Otherwise let

$$Y_\nu \subset X_k^{sm}$$

denote the collection of all subschemes constructed above, for all components y having the minimal coefficient m. For each ν denote by

$$X_\nu \to X$$

the blow-up of Y_ν. Then X_ν is smooth over V over the preimage of the smooth locus of Y_ν, and the blow-up is ramified there. It follows that for the elements of $\mathcal{E}$ which specialise in Y_ν the induced points in X_ν specialise in the smooth locus but in a component which has coefficient $> m$ in the divisor of α.

Now we can find a model $\tilde{X}$ which dominates all X_ν, and by Néron desingularisation we may assume that all elements of $\mathcal{E}$ specialise in $\tilde{X}^{sm}$. We claim that they only specialise in components with coefficient $> m$ in $div(\alpha)$: Namely for such a point $x \in \tilde{X}^{sm}(V')$ either the map

$$\tilde{X} \to X_\nu$$

or the map

$$\tilde{X} \to X$$

factor in a neighbourhood of x over the smooth locus of the range, and there the specialisation is contained in an irreducible component with coefficient $> m$. But the coefficient can only grow by pullback.

So finally the claim has been shown.

3. Differential Operators

If
$$f : X \to S$$
is a smooth map of schemes the sheaf
$$\mathcal{D}_{n,X/S}$$
of differential operators of order $\leq n$ is defined inductively as follows:

$\mathcal{D}_{n,X/S}$ is a subsheaf of the $\mathcal{O}_S$-linear endomorphisms of $\mathcal{O}_X$.
$$\mathcal{D}_{0,X/S} = \mathcal{O}_X$$
consists of scalar multiplications, and D has degree $\leq n$ iff the commutator $[D, f]$ with any $f \in \mathcal{O}_X$ has degree $\leq n-1$.

Equivalently if
$$\mathcal{I}_\Delta \subset \mathcal{O}_{X\times X}$$
denotes the ideal of the diagonal
$$X \subset X \times_S X,$$
then $\mathcal{D}_{n,X/S}$ is the dual of $pr_{1,*}(\mathcal{O}_{X\times X}/\mathcal{I}_\Delta^{n+1})$.

The union
$$\mathcal{D}_{X/S} = \bigcup_n \mathcal{D}_{n,X/S}$$
is a filtered algebra and a bimodule over $\mathcal{O}_X$. If X is (locally) étale over $\mathbb{A}^d_S$, with local coordinates
$$t_1, \dots, t_d,$$
and dual derivations
$$\partial_1, \dots, \partial_d,$$
then $\mathcal{D}_{X/S}$ is the free $\mathcal{O}_X$-module (for either $\mathcal{O}_X$-structure) in the divided power monomials
$$\prod_i \partial_i^{n_i}/n_i!,$$
Especially its associated graded is naturally isomorphic to the divided power algebra of the relative tangent-bundle $\mathcal{T}_{X/S}$.

Finally $\mathcal{D}_{X/S}$ admits a cocommutative coalgebra structure: The exists an algebra homomorphism
$$\lambda : \mathcal{D}_{X/S} \to \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}$$

(left $\mathcal{O}_X$-module structure)

such that

$$\lambda(D)(f \otimes g) = D(fg).$$

If

$$G \to S$$

is a smooth relative group-scheme the sheafs $\mathcal{D}_{n,G/S}$ are induced from S. Namely a differential operator is called right invariant if it commutes with right translations by elements of G. The right invariant differential operators $D_{n,G/S}$ of degree $\leq n$ form a vectorbundle on S. They admit the structure of a cocommutative Hopf-algebra. Their dual is the formal group $\hat{G}$:

Recall that if

$$\mathcal{I}_e \subset \mathcal{O}_G$$

denotes the ideal of the unit-section then

$$\mathcal{O}_G/\mathcal{I}_e^{n+1}$$

defines an $\mathcal{O}_S$-flat finite subscheme

$$G_n \subset G.$$

The multiplication maps $G_m \times G_n$ into G_{n+m} so the union

$$G_\infty = \bigcup_n G_n$$

is an ind-groupscheme contained in G. Furthermore the dual of the affine algebra of G_n is equal to $D_{n,G/S}$. Namely for l in this dual the associated

$$D \in D_{n,G/S}$$

acts on

$$f \in \mathcal{O}_G$$

as follows:

Consider the function $f(gh)$ on

$$G_n \times G.$$

It can be considered as a section of the constant sheaf $\mathcal{O}_{G_n}$ on G. Applying the linear form l on $\mathcal{O}_{G_n}$ defines $D(f)$.

Furthermore the multiplication and comultiplication on $\mathcal{O}_{G_n}$ correspond to comultiplication and opposite multiplication on $D_{n,G/S}$.

Now we return to our standard-situation: V is a discrete valuation-ring, K its field of fractions, G_K a smooth groupscheme over K, $\mathcal{E}$ a bounded and open family of Néron-points of G_K. The top differentials $\Omega^d_{G_K/K}$ can be trivialised by chosing either a left- or a right-invariant generator. The two differ by a character

$$G_K \to \mathbb{G}_m$$

which is the inverse of the determinant of the adjoint representation. This character takes unit-values on the elements of $\mathcal{E}$: Choose a quasiprojective extension X/V of G_K/K such that all elements of $\mathcal{E}$ extend to V'-points of X. The determinant of the adjoint representation defines a meromorphic function on X which is regular and nonzero on the generic fibre

$$X_K = G_K.$$

Thus it becomes a regular function after multiplication with a fixed power π^r, and the same for its inverse, and its valuation on Néron-points in $\mathcal{E}$ is bounded in size by r. As these Néron-points are closed under multiplication all these values must be units. An application:

In the previous section we have constructed a quasi-projective model X such that the elements of $\mathcal{E}$ extend to integral points of the smooth locus X^{sm}. Also we may assume that each irreducible component of the special fibre X_k^{sm} contains the specialisation of an element of $\mathcal{E}$. If we chose a generator of $\Omega^d_{G_K/K}$ at the origin we can extend it to a generator of $\Omega^d_{G_K/K}$ on G_K by either left or right translation, and obtain two meromorphic top differential forms α on X^{sm}. Then their divisors coincide. We may normalise the generator so that all irreducible components of X_k^{sm} have coefficient ≥ 0 in it, and that we have equality for at least one component for which the specialisations of elements of $\mathcal{E}$ are dense. Define

$$U \subset X^{sm}$$

as the union of G_K and these components. Thus $\Omega^d_{U/V}$ is free where a generator extends either the left-invariant or the right-invariant generator on G_K. Note that the construction of U commutes with base-extension

$$V \subset V_1.$$

Define a thick open subset

$$U' \subset U$$

as an open subset which contains the generic fibre G_K and is dense in U_k, and similar after base-extension. Then

Lemma 3. *Suppose $x \in G_K(K')$ is a Néron-point extending to $x \in X^{sm}(V')$. Then there exist thick open subsets*

$$U', U" \subseteq U_{V'}$$

such that multiplication by x on $G_{K'}$ (on the left or on the right) extends to an isomorphism

$$U' \cong U".$$

Proof. We may assume

$$V = V'.$$

Let

$$Z \subset U \times X^{sm}$$

denote the closure of the graph of multiplication by x. Then Z is flat over V, of relative dimension d. By dimension-count the first projection

$$Z_k \to U_k$$

is quasifinite over a thick

$$U' \subseteq U.$$

If

$$Z' \subseteq Z$$

denotes the preimage of U' then by Zariski's main theorem (see for example [8], Th. 4.1) the projection

$$Z' \to U'$$

factors as the product of an open immersion and a finite map. The latter is an isomorphism (generic fibre) so

$$Z' \to U'$$

is an open immersion. Now for any element y of $\mathcal{E}$ specialising in U' the pair (y, xy) lifts to an integral point of Z'. Thus Z_k is dense in U'_k, and we may assume that Z' is the graph of a regular map

$$U' \to X^{sm}.$$

The generator α of $\Omega^d_{G_K/K}$ is respected by this map. As α is regular on X^{sm} and a generator on U it follows that α must generate $\Omega^d_{X^{sm}/V}$ in the image and our map is étale. Again by Zariski's main theorem it is an isomorphism onto an open

$$U" \subset X^{sm}.$$

Furthermore α generates $\Omega_{U"/V}$ and the specialisations of elements of $\mathcal{E}$ are Zariski-dense in $U"_k$. Thus

$$U" \subset U.$$

Finally applying the same to multiplication by x^{-1} exchanges U' and $U"$, so $U"$ is thick. □

A variant of the proof also gives:

Lemma 4. *There exists a thick subset*

$$\tilde{U} \subset U \times U$$

such that the multiplication on G_K extends to a regular map

$$\tilde{U} \to U.$$

Its restriction to any subscheme obtained by fixing one factor is étale.

Proof. As before denote by

$$Z \subset U \times U \times X^{sm}$$

the closure of the graph of the multiplication map on G_K. There exists a thick open subset of $U \times U$ over which the first projection from Z is quasifinite, hence an open immersion. Its image is also thick ($\mathcal{E}$-points) so we have a regular map $\tilde{U} \to X^{sm}$. The pullback of α on X_{sm} is a d-form on $\tilde{U}$ whose restriction to any coordinate axis is either α or a unit times α. Especially it vanishes nowhere so α does not vanish on the image, and our map is smooth and even étale on coordinate axis. Finally the specialisations of $\mathcal{E}$-points are Zariski-dense on the image, so the map factors over U. □

Remark: If

$$Y \subset U_k \times U_k$$

denotes the complement of $\tilde{U}_k$ then Y has dimension $\leq 2d-1$. So we can find a thick

$$U' \subset U$$

such that the two projections from Y to U_k have fibres of dimension $\leq d-1$ over U'. It follows that this remains so if we replace $\tilde{U}$ by its intersection $\tilde{U}'$ with $U' \times U'$. That is the fibres of the projections $\tilde{U}' \to U'$ are themselves thick.

We now come to our main technical result:

Proposition 5. *The formal group $\hat{G}_K$ extends to a formal group $\hat{G}$ over V such that the (left) action of $\hat{G}_K$ on G_K extends to an action of $\hat{G}$ on U. This induces a trivialisation of the bundle $\mathcal{D}_{n,U/V}$ which becomes isomorphic to the trivial bundle dual to $\mathcal{O}_{G_n}$. Equivalently the action induces an isomorphism of formal schemes*

$$\hat{G} \times U \cong \widehat{U \times U}$$

(formal completion along the diagonal).

Proof. As before $D_{n,G_K/K}$ denotes the differential operators of degree $\leq n$ on G_K which are invariant under right translation. Let $D_{n,G/V}$ denote the V-submodule of differential operators which extend to U. Its formation commutes with base-change. We show that for any Néron-point $x \in \mathcal{E}$ specialising to U' the fibre of $\mathcal{D}_{n,U/V}$ in x is equal to the base-extension $D_{n,G/V} \otimes_V V'$. For this we may assume that $V = V'$.

Now the fibre obviously contains $D_{n,G/V}$. Conversely the multiplication-map extends to

$$\tilde{U} \subset U' \times U'$$

and the intersection

$$\tilde{U}_x = \{x\} \times U' \cap \tilde{U}$$

is still thick in U, and left multiplication by x maps it into $x\tilde{U}_x$ which is also thick (follows easily from lemma 4). If f is a section of $\mathcal{O}_U$ on an open subset of $x\tilde{U}_x$ its pullback by the multiplication-map is defined on an open subset of $\tilde{U}$. If D denotes a regular section of $\mathcal{D}_{n,U/V}$ in a neighbourhood of x apply $D \otimes 1$ to the pullback of f and restrict to

$$\{x\} \times \tilde{U}_x.$$

The result only depends on the value of D in x and defines a differential operator on $x\tilde{U}_x$ which is right translation-invariant on the generic fibre G_K. As $x\tilde{U}_x$ is thick it extends to all of U, and we have the other inclusion.

Thus $D_{n,G/V}$ generates $\mathcal{D}_{n,U/V}$ over all points of $\mathcal{E}$ specialising in U'. Hence the determinant of the induced map of vectorbundles does not vanish at these points, and by density it must be an isomorphism on all of U.

Now multiplication and comultiplication on the $D_{n,G/V}$ (the latter is integral because its base-extension to $\mathcal{D}_{n,U/V}$ is) define such operations on its dual $\mathcal{O}_{G_n}$ which give us the formal group $\hat{G}$. The claim about its operation on U is a reformulation of its definition via differential operators. $\square$

Remark. The same reasoning also gives a right action of $\hat{G}$ on U: The inverse on G_K extends by the usual argument to an isomorphism

$$U' \cong U"$$

of thick opens. Thus the left action on U' corresponds to a right action on $U"$ induced from a trivialisation of the sheaf of differential operators. This trivialisation extends to all of U.

4. The construction of the Néron-model

As before G_K is a smooth geometrically irreducible groupscheme, of relative dimension d, $\mathcal{E}$ a bounded and open set of Néron-points closed under multiplication. We have constructed a quasi-projective model X for G_K and an open subset

$$U \subset X^{sm}$$

with certain properties. Especially we have found a formal group $\hat{G}$ over V extending $\hat{G}_K$ which operates from the left and from the right on U, such that

$$\hat{G} \times U \cong \widehat{U \times U}.$$

Over K the map

$$G_K \times G_K \to G_K$$

which sends a pair (g, h) to $g^{-1}h$ is a smooth surjection. It defines an equivalence relation R_K on

$$G_K \times G_K$$

which is the equivalence relation defined by the diagonal left G_K-action on G_K^2. Denote by

$$R \subset U^4$$

its closure over V. Thus R is flat over V, of relative dimension $3d$. Furthermore R is invariant under all $\hat{G}$-operations which preserve R_K: As G_n is flat over V the product with it preserves closures.

This includes the following operations on quadruples

$$(g_1, h_1, g_2, h_2) \in U^4 :$$

- multiplying g_1 and h_1 by a common $g \in \hat{G}$ from the left,
- multiplying g_2 and h_2 by a common $g \in \hat{G}$ from the left,
- multiplying g_1 and g_2 by a common $g \in \hat{G}$ from the right,
- multiplying h_1 and h_2 by a common $g \in \hat{G}$ from the right.

We claim that all projections from R to three factors U^3 are open immersions. Since this holds over K it suffices to show that they are quasifinite. Also by known symmetries it suffices to treat the projection onto the first three factors.

For this consider

$$R_{\bar{k}} \subset U_{\bar{k}}^4$$

over the algebraic closure $\bar{k}$ of k. If $\hat{R}_{\bar{k}}$ denotes its formal completion in a point (g_1, h_1, g_2, h_2) then the local ring of this formal scheme has dimension $3d$. On the other hand using three of the $\hat{G}$-action it is isomorphic to the product of $\hat{G}^3$ with its intersection with

$$\{(g_1, h_1, g_2)\} \times U.$$

By reasons of dimensions the last factor has dimension zero, and this means the required quasifiniteness. As this holds at all $\bar{k}$-points we are done.

Now the two projections from R to

$$U \times U$$

are smooth. It follows that R is indeed an equivalence relation (by continuity), and the quotient

$$G = U \times U/R$$

exists as a separated algebraic space, smooth over V, and extends G_K. We shall see that G is a group but this still requires some arguments.

In more generality we can define a smooth separated algebraic spaces $\mathbb{G}_n$ as the quotient of U^{2n} under the equivalence relation which is the closure of the equivalence relation on G_K^{2n} defined by the map

$$G_K^{2n} \to G_K$$

which sends

$$(g_1, h_1, g_2, h_2, \ldots, g_n, h_n)$$

to

$$g_1^{-1} h_1 g_2^{-1} h_2 \cdots g_n^{-1} h_n.$$

As before one shows that the closure is indeed an equivalence relation contained in U^{4n}, and the projection to any partial product U^{4n-1} is an open immersion. Finally if

$$\Delta_U \subset U \times U$$

denotes the diagonal the first projection

$$U^{2n} \times \Delta_U \to U^{2n}$$

and the inclusion

$$U^{2n} \times \Delta_U \subset U^{2n+2}$$

define a map

$$\mathbb{G}_n \to \mathbb{G}_{n+1}$$

which extends the identity on G_K and is an open immersion. Finally the inverse on G_K extends to an automorphism of each $\mathbb{G}_n$, and the multiplication to a product

$$\mathbb{G}_m \times \mathbb{G}_n \to \mathbb{G}_{m+n}.$$

For any $x \in \mathcal{E}$ the multiplication by x defines after base-extension to V' an isomorphism

$$U' \cong U"$$

of thick open subschemes of $U_{V'}$. It follows that x extends to a V'-point of $\mathbb{G}_1$. Furthermore the set of these points is closed under multiplication and inverses, and their reductions are scheme-theoretically dense in the special fibre $\mathbb{G}_{n,k}$ (for each n) because the reductions are dense in U_k.

Finally the multiplication on G_K extends to

$$m : \tilde{U} \to U$$

for some thick

$$\tilde{U} \subset U \times U.$$

In fact we can chose a thick

$$U' \subset U$$

such that

$$\tilde{U} \subset U' \times U'$$

and such that the two projections from $\tilde{U}$ to U' have thick fibres. Now the second projection

$$\tilde{U} \to U'$$

is surjective, and the map

$$(pr_1, m) : \tilde{U} \to U' \times U$$

is an open immersion which factors over the quotients to define an open immersion $U' \to \mathbb{G}_1$ which extends the identity on G_K. We claim that on the geometric special fibre $\mathbb{G}_{1,\bar{k}}$ the image of $U'_{\bar{k}}$ meets each connected component:

For any $x \in \mathcal{E}$ multiplication by x defines over V' an isomorphism

$$U'_1 \cong U'_2$$

of thick opens in $U'_{V'}$. So if we chose a suitable point

$$z \in U'_1(\bar{k})$$

in a fixed connected component of $U_{\bar{k}}$ the product $z^{-1}U'_2$ contains the specialisation of x, and the product meets its connected component if we chose z arbitrarily but in a fixed component. However as the specialisations of elements of $\mathcal{E}$ are dense they meet all components. Conclusion: In the group

$$\bigcup_n \pi_0(\mathbb{G}_{n,\bar{k}})$$

any element is the difference of the class of z and an element of $\pi_0(U'_{\bar{k}})$, and this gives our claim. It implies also that $U'_{\bar{k}}$ is Zariski-dense in each $\mathbb{G}_{n,\bar{k}}$ and so any element in the union of all $\mathbb{G}_{n,\bar{k}}$ is the difference of two elements in $U'_{\bar{k}}$. Thus the union is

$$G = \mathbb{G}_1,$$

and this is a group.

To show that G is a scheme we can make U' smaller and assume that it is affine (and $U_{\bar{k}}$ still dense), as U is quasiprojective. Of course the generic fibre U'_K ceases to be G_K. The inclusion

$$U' \to G$$

is an affine map so the complement D is a divisor in G, flat over V. Then

$$\mathcal{L} = \mathcal{O}(D)$$

is ample so G is quasiprojective: It suffices to show this after base-change to the strict henselisation of V. If

$$g_i \in G^\circ(V)$$

denote sections in the connected component of the identity, with product the neutral element e, then the tensorproduct of all g_i-translates of some power of $\mathcal{L}$ is isomorphic to a power of $\mathcal{L}$, by the theorem of the cube (it holds in the generic fibre, which implies it over V). This gives plenty of sections of powers of $\mathcal{L}$ such that the complement of their divisor is affine.

The final claim in the main theorem (extending maps from S) is well known: Let

$$Z \subset S \times G$$

denote the closure of the graph of the generic map

$$f_K : S_K \to G_K.$$

the projection

$$Z_k \to S_k$$

is dominant (points in $\mathcal{E}$ which lift) and quasifinite over a thick open subset. It follows that f_K extends to a regular map

$$f : S^\circ \to G$$

on a thick open

$$S^\circ \subseteq S.$$

If u is an affine neighbourhood of the identity in G the map

$$f(s)^{-1} f(t) : S^\circ \times S^\circ \to U$$

extends to a neighbourhood of the diagonal of S, as the pullback of any regular function on U is regular precisely on the complement of a divisor on $S \times S$. This divisor meets the diagonal only on the complement of S° which has codimension ≥ 2, so does not intersect the diagonal at all.

As for any $s \in S$ we can find a $t \in S^\circ$ such that (s,t) lies in this neighbourhood of the diagonal we can extend f to a neighbourhood of s.

5. Semistable reduction

We give a geometric proof for the semistable reduction theorem. We start with a technical result which is a weaker version of the main result in [3]:

Proposition 6. *Suppose V is a discrete valuation-ring, with field of fractions K, and X a flat V-scheme of finite type such that X_K is geometrically reduced, and the special fibre X_k is non-empty. Then there exists a finite separable extension K' of K such that for all valuation-rings $V' \subset K'$ dominating V the normalisation X' of $X_{V'}$ has geometrically reduced special fibre $X'_{k'}$.*

Remarks. a) We may replace X by any Y such that Y_K is smooth and there exists a map

$$Y \to X$$

which is dominant on the special fibre: Assume X and Y are normal and Y_k is geometrically reduced. Every generic point of an irreducible component of X_k is the image of such a point for Y_k. Considering the induced map of discrete valuation-rings easily gives the assertion. For example we may remove from X the closure of the complement of the smooth locus of X_K. So we may (and will) assume that X_K is smooth.

b) The normalisation is always finite over $X_{V'}$: This holds over the completion $\hat{V}'$. However as the generic fibre X_K is smooth this normalisation is already defined over V'. This shows also that we may replace V by its completion $\hat{V}$, as all separable extensions of the fraction-field of $\hat{V}$ are induced from separable extensions of K (approximate the coefficients of a separable polynomial).

c) It suffices to show the assertion for one V' and one irreducible component of $X'_{k'}$: Namely we may assume that X_k is irreducible and that K' is Galois over K. Then all choices are conjugate under the Galois-group.

Proof. We start with a number of reduction steps:

a) We may assume that V is strictly henselian, and that the residue-field k is perfect: If u is a unit in V whose reduction modulo π does not lie in k^p ($p > 0$ the characteristic of k) the equation

$$T^{p^2} + \pi T = u$$

defines an extension V', separable over K, such that k' is obtained from k by adjoining a p^2-th root of u. Applying this procedure to a p-base of k and repeating we find a filtering increasing family of

$$V_\alpha \supset V,$$

with fraction-field K_α separable over K, and still uniformiser π. Their union is then a discrete valuation-ring V with perfect residue-field k'. If a Y with the desired properties exists over V' it already exists over some V_α, etc..

b) Also we may replace X by any open subscheme with non-empty special fibre, and assume that

$$X = Spec(R)$$

is affine. We can find elements $T_1, \ldots, T_d$ such that after passing to an open subscheme they define a quasifinite map

$$X \to \mathbb{A}^d_V$$

which is étale over K:

First choose the T_i such that they induce a finite map over k (Noether normalisation). Then modify them by adding elements divisible by π such that the differentials dT_i become linearly independant in the generic points of X_K. Then they form a basis of $\Omega_{X_K/V}$ on a dense open subset of X_K. Remove from X the closure of its complement.

c) We make induction over the relative dimension d of X/V. If $d > 1$ replace X by an open affine subscheme such that there exists a flat homomorphism

$$X \to \mathbb{A}_V^{d-1},$$

smooth on the generic fibre. Then apply induction to the induced scheme (of relative dimension one) over the localisation W of $\mathbb{A}^{d-1}$ in its generic point in characteristic p. By induction there exists a $W' \supset W$ where we obtain geometrically reduced fibres. $Spec(W')$ spreads out to a quasifinite, étale over K

$$Z \to \mathbb{A}^{d-1}.$$

Then apply induction to Z. Thus it suffices to treat relative curves ($d = 1$).

d) Choose as before a quasifinite

$$X \to \mathbb{A}_V^1,$$

étale over K. We may replace X by an open subscheme of the Galois-hull of this cover. The induced map on henselisations in generic points of special fibres is finite. Passing to étale neighbourhoods we may assume that we have a finite Galois-covering

$$X \to Z,$$

with Z smooth over V (of relative dimension one), and the special fibres X_k, Z_k irreducible. Replacing Z by a finite étale covering of an open subscheme we may assume that the extension of residue-fields (in generic points of the special fibre) is purely inseparable. Furthermore adjoining an n-th root of π with n prime to p we may assume that the ramification index in these points is a power of p (Abhyankhar's lemma). Then our covering is totally ramified at these generic points, and the order of G is a power of the residue characteristic p (If $p = 0$ we are finished at this point). As the group is a successive central extension of $\mathbb{Z}/(p)$ we make induction over the degree and assume that it is p. Also we may assume that Z_k is geometrically irreducible over k.

e) Now replace V by finite extensions V', X by the normalisation of $X_{V'}$, and Z by $Z_{V'}$. If for some such extension the special fibre $X'_{k'}$ is geometrically reduced we are done. Otherwise it always has one irreducible component with multiplicity p and function-field equal to that of $Z_{k'}$.

In the following we consider discrete valuation-rings $W \supset V$ dominating V, of essentially finite type, such that the fraction-field L of W is separable over K. Furthermore we assume that the residue-field $\tilde{k}$ of W is separable over k, of transcendence degree one, and that k is algebraically closed in it. Then $\Omega_{W/V}$ is a finitely generated W-module. In fact it is generated by the derivatives

$$d\Pi, dt,$$

where Π is a uniformiser of W and t maps to a separating transcendence-basis of the residue field $\tilde{k}/k$. Especially for an extension

$$W_1 \subset W_2$$

with separable residue-field extension the relative differentials Ω_{W_2/W_1} can be generated by one element. Together with the following simple observation this turns out to be surprisingly powerful:

Lemma 7. *Suppose*

$$W_1 \subset W_2$$

is an extension, with

$$L_1 \subset L_2$$

separable. Then the sequence

$$(0) \to \Omega_{W_1/V} \otimes_{W_1} W_2 \to \Omega_{W_2/V} \to \Omega_{W_2/W_1} \to (0)$$

is exact, and Ω_{W_2/W_1} *is torsion.*

Proof. The map is a relative complete intersections. That is there exists an integer n, n elements

$$f_1, \ldots, f_n \in W_1[T_1, \ldots, T_n],$$

and a localisation of $W_1[T_1, \ldots, T_n]$ such that W_2 is the quotient of this localisation under the ideal generated by the regular sequence $f_1, \ldots, f_d$. Thus $\Omega_{W_2/V}$ is the quotient

$$\Omega_{W_2/V} = \Omega_{W_1/V} \otimes_{W_1} W_2 \oplus \bigoplus W_2 dT_i/(df_1, \ldots, df_n).$$

As the extension of fraction-fields is separable the matrix of derivatives $\partial f_i/\partial T_j$ becomes invertible over L_2, and the assertion follows. $\square$

Now take as W_1 the localisation of Z in the generic point of Z_k and for W_2 the local ring of X' in a generic point of $X'_{k'}$. Then $\Omega_{W_1/V}$ is a free W_1-module of rank one. Furthermore as the extension of residue-fields is trivial Ω_{W_2/W_1} is generated by $d\Pi$, Π a uniformiser of W_2.

$$F(\Pi) = \Pi^e + a_1\Pi^{e-1} + \cdots + a_e = 0$$

is its minimal equation over W_1. Finally Ω_{W_2/W_1} is isomorphic to

$$W_2/F'(\Pi)$$

where F' denotes the T-derivative of Π.

Now the torsion in Ω_{W_2/W_1} injects into

$$\Omega_{W_2/W_1} = \Omega_{W_2/V}/\Omega_{W_1/V} \otimes_{W_1} W_2.$$

So either it surjects onto the quotient or it does not.

Assume first that the second alternative holds for some V'. If we replace V' by a bigger ramified V", to obtain a new W_3, the inclusion

$$\Omega_{W_2/V} \otimes_{W_2} W_3 \subset \Omega_{W_3/V}$$

induces an isomorphism on torsion. Namely otherwise we use the fact that the submodules of the cyclic W_3-module Ω_{W_3/W_1} are linearly ordered, indexed by their length. So we obtain a torsion-class in $\Omega_{W_3/V}$ whose image in Ω_{W_3/W_2} lies in the image of $\Omega_{W_2/W_1} \otimes_{W_2} W_3$ but not in the image of the torsion in $\Omega_{W_2/V} \otimes_{W_2} W_3$, and thus also not in the image of $\Omega_{W_2/V} \otimes_{W_2} W_3$ itself. A contradiction.

Thus the torsion in $\Omega_{W_3/V}$ is annihilated by some fixed power of π'. Choose V" so that this power of π' strictly divides the different of $V"/V'$. For example adjoin to V' a root of

$$T^{p^a} + \pi'^{,b} T = \pi'$$

with sufficiently large a, b. Then the tensorproduct $V" \otimes_V W_1$ is a discrete valuation-ring and its Ω injects into $\Omega_{W_3/V}$. Especially this also holds for $\Omega_{V"/V}$ which however is torsion and not annihilated by our fixed π'-power. Thus a contradiction and the other alternative holds.

This means that the torsion surjects onto Ω_{W_2/W_1}, that is canonically

$$\Omega_{W_2/V} = \Omega_{W_1/V} \otimes_{W_1} W_2 \oplus \Omega_{W_2/W_1}.$$

We apply this as follows: Choose a

$$t \in W_1$$

which induces a separating transcendence basis on the residue-field. Then the derivation

$$D = \partial/\partial t$$

extends uniquely to a compatible system of derivations on all W_2.

We can do even better: The differential operators

$$D_n = D^n/n!$$

generate (with a suitable definition in characteristic p) the differential operators on the affine line and thus define V-linear differential operators on W_1.

By étaleness they extend to a compatible system of differential operators on $W_2 \otimes_{V'} K'$. We claim that they respect W_2:

We use induction on n, so assume the assertion holds for D_a with $a < n$. By the rule

$$D_n(fg) = \sum_{a+b=n} D_a(f)D_b(g)$$

$D_n$defines a compatible system of $W_1 \otimes_V V'$-linear derivations

$$W_2 \to W_2[1/\pi]/W_2,$$

that is a compatible system of linear maps

$$\Omega_{W_2/W_1\otimes_V V'} \to W_2[1/\pi]/W_2.$$

However such a compatible system vanishes because for each V' there exists a $V"$ (with W_3 the appropiate normalisation) such that

$$\Omega_{W_2/W_1\otimes_V V'} \to \Omega_{W_3/W_1\otimes_V V"}$$

is zero:

Namely chose $V"$ such that different of $V"/V'$ annihilates $\Omega_{W_2/W_1\otimes_V V'}$. Then in the cyclic W_3-module $\Omega_{W_3/W_1\otimes_V V'}$ the image of $\Omega_{W_2/W_1\otimes_V V'}$ is contained in the submodule generated by $\Omega_{V"/V'}$ (by consideration of length), thus maps to zero in the quotient $\Omega_{W_3/W_1\otimes_V V"}$.

Thus we know that all differential operators on W_1/V extend to all W_2. An equivalent description:

Spread out $Spec(W_1)$ to a smooth scheme Z/V, Z_k geometrically irreducible, together with a finite flat $\mathbb{Z}/(p)$-equvariant cover

$$X \to Z,$$

smooth over K. That the differential operators on Z extend to X means that the two pullbacks of X to $Z \times_V Z$ coincide on the formal completion along the diagonal. Choose a V-point

$$z \in Z(V).$$

Restricting to $Z \times \{z\}$ we get that the restriction of X to the formal completion of Z in z is isomorphic to the constant cover given by the fibre of X in z. As we may assume that V is complete thus excellent this restriction is still normal, so the constant cover must be normal as well. Pass to a finite extension of V to make it trivial over K and thus over V itself. Then X becomes étale over the completion of Z at z, thus the relative discriminant is a unit there and then a unit on Z, thus X is étale over Z which contradicts our assumptions. $\square$

Theorem 8. *Suppose G_K is an abelian variety over K, of dimension d. There exists a finite separable extension K' of K such that the Néron model of $G_{K'}$ has semistable reduction. That is the connected component of its special fibre is an extension of an abelian variety by a torus.*

Proof. We may assume that V is strictly henselian. The connected component G_k° of the special fibre of the Néron-model G of G_K is an extension

$$(0) \to H \to G_k^\circ \to A \to (0),$$

with H a linear algebraic group and A an abelian variety. Furthermore H is an extension of a torus T by a unipotent group. Denote by a, t, u the dimensions of these, so

$$a + t + u = d.$$

If we replace V by a finite extension V' the new Néron-model G' admits a map

$$G \otimes_V V' \to G'.$$

It induces a map on connected components

$$G_k^\circ \otimes_k k' \to G_{k'}^{\circ,\prime}$$

which induces maps on abelian and torus components A and T. By considering n-torsion-points with n prime to the characteristics it follows that on these components the induced maps have finite kernels, so the dimensions increase. Pass to a finite extension such that a and t take their maximal values. Then under base-extensions the induced maps on A and T become isogenies. This also holds if we replace G_K by an isogeneous abelian variety.

Choose an ample line-bundle on the Néron-model G of G_K. Then its restriction to G_k° is still ample and pullback of a line-bundle $\mathcal{L}_A$ on A. The latter is still ample:

If the associated map

$$A \to A^t$$

has positive-dimensional kernel we find a positive dimensional abelian subvariety

$$A' \subset A$$

such that our line-bundle lies in $Pic^0(A')$. Then for an ample line-bundle $\mathcal{M}$ on A' the tensorproduct

$$\mathcal{L}^{\otimes n} \otimes \mathcal{M}^{\otimes -1}$$

has no nonzero global sections on the preimage of A' in G_k°, for all integers n (this contradicts ampleness):

If we had such a section there would be also one fixed by the unipotent radical of H. But it would give nonzero sections on A' of the tensor-product of $\mathcal{M}^{\otimes -1}$ with elements of $Pic^0(A')$, which cannot happen.

So $\mathcal{L}_A$ is non-degenerate, and by considering global sections as before it must be ample. This property is independant of the choice of the extension $\mathcal{L}$ of $\mathcal{L}_K$, and persists also after extending V.

Now we can divide G_K by a maximal isotropic subgroup and assume that $\mathcal{L}_K$ defines a principal polarisation on G_K, with ample $\mathcal{L}_A$. Furthermore we assume that for an odd prime

$$l > d!$$

and different from p the l-division-points $A[l]$ are K-rational, and that the Weil-pairing defined by $\mathcal{L}_A$ on $A[l]$ is non degenerate.

For the next considerations assume only that G is a smooth separated group-scheme over V extending G_K such that all l-division-points of G_K extend to V-points in G. For example we could form the Néron-model of G_K and denote by G its open and closed subscheme whose generic fibre is G_K but whose special fibre is the union of those connected components which contain an l-division-point (that is whose order in the π_0 of the special fibre is one or l). In the following we use the theory of biextensions from [4], Exp. 7,8 or [5], Ch. II.

Denote by

$$D \subset G$$

the closure of the theta-divisor on G_K. Then $\mathcal{O}(D)$ is a line-bundle on G and over K it defines a symmetric biextension of $G_K \times G_K$ by $\mathbb{G}_m$. Its direct image gives over V a symmetric biextension of $G \times G$ by $\mathbb{G}_m \times \mathbb{Z}$ (the direct image of $\mathbb{G}_m$). The induced biextension by $\mathbb{Z}$ is pullback of a biextension over $\pi_0(G_k) \times \pi_0(G_k)$. Its base-extension to $\mathbb{Q}$ splits canonically so it is the boundary of a unique symmetric bilinear form

$$\pi_0(G_k) \times \pi_0(G_k) \to \mathbb{Q}/\mathbb{Z}$$

which is precisely Grothendieck's bilinear pairing. After passing to a finite ramified extension of V this pairing becomes trivial. That is there exists a section which is additive in both partial addition laws. This means the following:

Consider the line-bundle

$$\mathcal{M} = m^*\mathcal{L} \otimes pr_1^*\mathcal{L}^{\otimes -1} \otimes pr_2^*\mathcal{L}^{\otimes -1}$$

on $G \times G$. Then there exists a divisor E on $G \times G$ which is a linear combination of connected components of G_k such that $\mathcal{M}(E)$ defines a biextension which extends the biextension on the generic fibre. It follows that after passing to a ramified extension we may change the extension $\mathcal{L}$ such that it satisfies the theorem of the cube:

Namely on G^3 the alternating tensorproduct of the pullbacks of $\mathcal{L}$ under all partial additions is associated to a divisor F with support in the special fibre. It is given by an integer valued function $\psi(a,b,c)$ on

$$\pi_0(G_k) \times \pi_0(G_k) \times \pi_0(G_k)$$

which is symmetric and vanishes if one of the arguments is 0. Similarly E defines a symmetric function $\phi(a,b)$ with

$$\psi(a,b,c) = \phi(a+b,c) - \phi(a,c) - \phi(b,c).$$

That the left side is symmetric implies that

$$\phi(a,b) + \phi(a+b,c) = \phi(a,b+c) + \phi(b,c).$$

That is ϕ is a cocycle defining an extension of $\pi_0(G_k)$ by $\mathbb{Z}$. Passing to a ramified extension of V we can trivialise this which implies the claim.

We derive as usual that there exists a central extension $\mathcal{G}$ of $G[l]$ by $\mathbb{G}_m$ which acts equivariantly on $\mathcal{L}^{\otimes l}$ (see [5], Ch. 5). The commutator-pairing on $\mathcal{G}$ induces the Weil-pairing on $G[l]$. Furthermore the extension reduces canonically to an extension by μ_l. Namely the elements of l-power order in $\mathcal{G}$ form a subgroup (using that l is odd).

Now $\mathcal{G}$ is the unique extension of its generic fibre $\mathcal{G}_K$, and $\mathcal{L}^{\otimes l}$ extends its generic fibre $\mathcal{L}^{\otimes l}$ as a $\mathcal{G}$-equivariant line-bundle. Furthermore the global sections $\Gamma(G, \mathcal{L}^{\otimes l})$ form a $\mathcal{G}$-invariant V-lattice in the induced K-vectorspace. By the representation-theory of $\mathcal{G}$ ([5], Ch. 5, Th.2.5.5) this lattice is unique up to scalars, and its reduction modulo π is an irreducible representation of $\mathcal{G}_k$. There exists an extension $\mathcal{M}$ of $\mathcal{L}_K^{\otimes l}$ (possibly different from $\mathcal{L}^{\otimes l}$) which is over an open dense subset of G_k generated by this lattice. $\mathcal{M}$ is $\mathcal{G}$-equivariant, and we claim that it is globally generated by our lattice:

We may assume that V is strictly henselian. Choose a nonzero global section

$$\vartheta \in \Gamma(G, \mathcal{L}).$$

Its vanishing order on the component labeled by $a \in \pi_0(G_k)$ is some integer $\mu(a)$, that is $\pi^{-\mu(a)}\vartheta$ is regular on this component and does not vanish generically. For any l-tuple of points

$$x_1, \ldots, x_l \in G(V)$$

with vanishing sum the product of all x_i-translates of ϑ can be identified with a global section of $\mathcal{L}^{\otimes l}$, and if the x_i map to

$$b_i \in \pi_0(G_k)$$

this global section is on the a-component divisible by π to the exponent

$$\sum_i \mu(a + b_i).$$

It follows that if we define

$$\lambda(a) = inf \sum_i \mu(a + b_i),$$

the infimum being taken over all sequences $b_i \in \pi_0(G_k)$ with sum zero, then these global sections generate

$$\pi^{\lambda(a)} \mathcal{L}^{\otimes l}$$

over the a-component. Thus the assertion if we define $\mathcal{M}$ to be equal to this on the a-component. Furthermore $\mathcal{M}$ is $\mathcal{G}$-equivariant because if the sum of l V-points x_i vanishes the same holds if we translate them by a common l-torsion-point y. That is $\lambda(a)$ is invariant under translation by l-torsion.

Finally we note that to define $\mathcal{M}$ it is not necessary to extend V: Consider the irreducible $\mathcal{G}_K$-representation

$$\Gamma(G_K, \mathcal{L}^{\otimes l}).$$

Choose in it a $\mathcal{G}$-invariant V-lattice. This is unique up to scalars, and the representation of $\mathcal{G}_k$ on its reduction modulo π is still irreducible. Then the lattice generates a line-bundle $\mathcal{M}$ on G which is $\mathcal{G}$-equivariant and extends $\mathcal{L}_K^{\otimes l}$.

Extending V itself is only necessary to get a cubic extension of $\mathcal{L}$.

Now choose a $\mathcal{G}$-invariant V-lattice as before and denote by X the closure of G_K in the corresponding projective space. The construction of X commutes with base-change, and the inclusion of G_K extends to a regular map

$$G \to X.$$

Especially this holds for the Néron-model G which is the universal case. Of course this map lifts to the normalisation

$$Y = X^{norm}$$

which need not be invariant under base-change. However by the proposition we may extend V and assume that the fibres of Y are geometrically reduced. The smooth locus Y^{sm} is then dense in the special fibre Y_k. Also by the Néron property the isomorphism $X_K \cong G_K$ extends to a map into the Néron-model

$$Y^{sm} \to G.$$

The composition with the map into Y is the identity on X_K and thus the identity, which implies easily that Y^{sm} is isomorphic to an open subscheme

$$U \subset G.$$

Because of the $\mathcal{G}$-equivariance U is $G[l]$-stable and nonempty. Also the irreducible components of Y_k correspond to the connected components of G meeting U, and contain at least one full $G[l]$-orbit.

The stabiliser of a component in $G_k[l]$ is $G_k^\circ[l]$ which is an extension of $A[l]$ by $T[l]$ and has order l^{2a+t}. Thus X_k^{sm} has at least

$$l^{2d-2a-t} = l^{t+2u}$$

irreducible components Y, and the degree of each such component is at most

$$d!l^{d-t-2u} = d!l^{a-u}.$$

Now each such component admits an action of

$$E = G_k^\circ[l]$$

and the induced $\mathbb{G}_m$-extension $\mathcal{H}$ acts equivariantly on $\mathcal{M}$. If $E^\perp$ denotes the perpendicular of E in the symplectic space $G_k[l]$ then the quotient

$$E/E \cap E^\perp$$

carries a non-degenerate symplectic form and thus has even dimension 2δ, and any representation of $\mathcal{H}$ on which the central $\mathbb{G}_m$ acts by a power prime to l has dimension divisible by l^δ. This applies to the cohomology-groups

$$H(Y, \mathcal{M}^{\otimes j})$$

for $0 < j \leq d$, thus the corresponding Euler-characteristics are divisible by l^δ, and the same applies to the degree of Y (using repeated differences). By the upper bound for that degree (which is nonzero) we obtain (using $l > d!$)

$$\delta \leq a - u.$$

Now the symplectic form on $G[l]$ is given by the biextension associated to the line-bundle $\mathcal{L}$ on G (which exists after extending V). Its restriction to G_k° is pullback of a line-bundle $\mathcal{L}_A$ on A, and the symplectic pairing on W is the pullback of the symplectic pairing on $A[l]$ defined by $\mathcal{L}_A$. If $u > 0$ this pairing cannot non-degenerate because otherwise $\delta = a$. However we assumed that $\mathcal{L}_A$ defines a non-degenerate Weil-pairing on $A[l]$, a contradiction.

For clarity we remark we had this non-degeneracy first for one V and G. Then we passed to a quotient and extended V. In both operations we obtained a map of Néron-models

$$G \to G'$$

such that the line-bundle $\mathcal{L}$ on G could be chosen as the pullback of the corresponding line-bundle on G'. This map induces isomorphisms on $A[l]$, hence the assertion. $\square$

6. Complements

Any smooth group-scheme G_K over K extends to a smooth G over V: Choose an open embedding of G_K into a projective V-scheme X. By Néron desingularisation we may assume that the unit section of G_K extends to a V-point in the smooth locus $U = X^{sm}$. Consider the closure Z of the graph of the multiplication-map

$$U_K \times U_K \to X.$$

The projection

$$Z \to U \times U$$

has connected fibres and admits a section over $\{e\} \times U$. Its image is a closed subscheme Z_1 contained in the full preimage Z_2 of $\{e\} \times U$. Over K these two coincide so the ideal defining Z_1 in Z_2 is annihilated by a power π^r.

Now define

$$\tilde{U} \to U$$

as the open affine subscheme of the the blow-up of the ideal generated by π^{r+1} and be the ideal of $\{e\}$ where π^{r+1} generates this ideal. If $\tilde{Z}$ denotes the new graph-closure over $\tilde{U} \times U$ we get new subschemes

$$\tilde{Z}_1 \subseteq \tilde{Z}_2$$

mapping to the previous. Assume W is a discrete valuation-ring dominating V and

$$(x_1, x_2) \in \tilde{U}(W) \times \tilde{U}(W)$$

is a point which lifts to points in Z and $\tilde{Z}$. The ideal defining Z_2 in Z becomes divisible by π^{r+1} on $\tilde{Z}$, so the pullback to W of the ideal defining Z_1 is divisible by π. That means (x_1, x_2) specialises to a point of Z_1 and especially its projection to X is equal to its second coordinate in $U \times U$. This means that the fibres of

$$\tilde{Z} \to \tilde{U} \times U$$

consist of one point, so this is an open embedding by Zariski's main theorem and thus an isomorphism by properness. In other words the multiplication on G_K extends to a regular map

$$\tilde{U} \times U \to U.$$

Now chose local coordinates S_i near e on U, vanishing at e (i.e. they define an étale map into affine space). Then the elements

$$T_i = S_i/\pi^{r+1}$$

are coordinates on $\tilde{U}$. The multiplication map is defined in these coordinates by powerseries $f_i(S, T)$ with

$$f_i(0, T) = \pi^{r+1} T_i,$$

$$f_i(S, 0) = S_i.$$

The f_i have the property that modulo each finite power π^s the coefficients of each S-monomial are polynomials in T. If we pullback to $\tilde{U} \times \tilde{U}$ we may divide also the S_i by π^{r+1}. Then all terms involving one of them become divisible by π^{r+1}, as do the remaining ones only involving T's. That is we may divide the f_i by π^{r+1} and obtain a regular map (first on π-adic formal schemes)

$$\tilde{U} \times \tilde{U} \to \tilde{U}.$$

That is $\tilde{U}$ is our desired extension.

If G_K is an abelian variety we may chose for $\mathcal{E}$ the class of all Néron-points and obtain the usual Néron-model. For the multiplicative group we could similarly chose all Néron-points of valuation zero. Of course we may also choose 1-units, etc.. For the additive group the choice of $\mathcal{E}$ is very relevant because the family of all Néron-points is not bounded.

References

[1] M. Artin, *Néron models,* Arithmetic Geometry, edited by G. Cornell and J.H. Silverman. Proc. Storrs 1984, Springer-Verlag, Berlin, 1986.

[2] S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron Models,* Springer-Verlag, Berlin, 1990.

[3] S. Bosch, W. Lütkebohmert and M. Raynaud, *Formal and rigid geometry IV. The reduced fibre theorem,* Invent. math. **119** (1995), 361–398.

[4] A. Grothendieck, *Groupes de Monodromie en Géometrie Algébrique* (SGA7,I), Springer Lecture Notes **288**, 1972.

[5] L. Moret-Bailly, *Pinceaux de variétés abéliennes,* Astérisque **129**, SMF, Paris, 1985.

[6] A. Néron, *Modèles'minimaux des variétés abeliennes sur les corps locaux et globaux,* Publ. Math. IHES **21** (1964).

[7] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes,* Lecture Notes **119**, Springer-Verlag, 1970.

[8] M. Raynaud, *Anneaux locaux henséliens,* Lecture Notes **169**, Springer-Verlag, 1970.

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Leonardo da Vinci Lecture held on May 3, 2007
Received: September 2007